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Discretely holomorphic parafermions in lattice Z_N models

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Abstract

We construct lattice parafermions—local products of order and disorder operators—in nearest-neighbor Z_N models on regular isotropic planar lattices and show that they are discretely holomorphic, that is they satisfy discrete Cauchy–Riemann equations, precisely at the critical Fateev–Zamolodchikov (FZ) integrable points. We generalize our analysis to models with anisotropic interactions, showing that, as long as the lattice is correctly embedded in the plane, such discretely holomorphic parafermions exist for particular values of the couplings which we identify as the anisotropic FZ points. These results extend to more general inhomogeneous lattice models as long as the covering lattice admits a rhombic embedding in the plane.

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1. Introduction

Z_N models, as the simplest statistical mechanics models with discrete global symmetries, have been studied for many years. They are interesting, both as lattice models and in the continuum limit as quantum field theories, because they exhibit semi-locality properties, such as fractional spin [1], which arise in various other domains of physics. The simplest examples of these models are the well-known Ising model ($N = 2$), three-state Potts model ($N = 3$) and Ashkin–Teller model ($N = 4$), all exactly solvable to some extent. The phase diagram of the nearest-neighbor Z_N model in two dimensions for $N = 5, 6, 7$ was elucidated in [2, 3], but in the general case it is still not known. Although these lattice models for $N > 4$ have complicated phase diagrams with several critical surfaces, Fateev and Zamolodchikov [4] showed that there are some points (hereinafter referred to as FZ points) in the critical surface at which these models are solvable, in the sense that they satisfy generalized star-triangle relations. Moreover, there is strong evidence that the scaling limit at these points corresponds to Zamolodchikov and Fateev's parafermionic conformal field theory [5]. In this paper, we give further evidence for this connection. In particular we find lattice candidates for the holomorphic parafermions of the continuum model and show that these satisfy a discrete version of the Cauchy–Riemann equations. We also extend this result to models with anisotropic interactions. We show that as long as the lattice is correctly embedded in the plane, we can again construct discretely holomorphic parafermions. We may use this to conjecture the location of the FZ critical surface of these anisotropic Z_N lattice models. In fact, our methods can be extended to any non-uniform lattice of the Baxter type [6], thanks to a result of Kenyon and Schlenker [7] about embedding such lattices in the plane.

Identifying discretely holomorphic objects in lattice models is an important step in showing that the scaling limit of suitably defined curves in these models is described by Schramm–Loewner evolution (SLE) [8, 9]. Suitable candidates for SLE curves in Z_N models have been suggested by Santachiara [10] (see also Gamsa and Cardy [11] for the case $N = 3$). We expect that our results will be the first step in showing that this is indeed the case.

This paper is organized as follows. In the second section we briefly review some related properties of Z_N models and the corresponding parafermionic conformal field theories. We define the disorder variables on the isotropic square lattice, similarly to the case of the Z_N clock models [1] and the Potts model [12]. We then define lattice parafermions as products of neighboring order and disorder variables, with a suitable phase factor. Using the Boltzmann weights of the model at the FZ point, we then show that there are local linear relations between parafermions, which are equivalent to discrete holomorphicity. This is our first main result.

In section 3, we consider lattices with anisotropic interactions and show that there are special points, which we identify with anisotropic FZ points, at which there is a certain embedding of the lattice on the plane for which discrete holomorphicity is again recovered. We further extend this to the case of Baxter lattices [6], where we show that the existence of a rhombic embedding [7] for such lattices allows us simply to generalize our results. It should be mentioned that the importance of rhombic embeddings (and the related isoradial embeddings) was central to the work of Duffin [13], Mercat [14] and Bobenko *et al* [15] in the general theory of discrete holomorphy. Similar ideas have been used in other lattice models by Kenyon [16] and Bazhanov *et al* [17].

2. Holomorphic parafermions in Z_N model

The general Z_N model on an arbitrary graph can be defined by associating with every node of the graph a variable s_r which takes values in the set $\omega^q, q = 0, 1, \ldots, N-1$, where $\omega = \mathrm{e}^{2\pi\mathrm{i}/N}$. Each configuration occurs with probability

$$Z^{-1} \prod_{(rr')} W(r, r'), \tag{1}$$

where the product is over all the edges (rr') of the graph and Z is the partition function. The weights $W(r, r')$ take the general form

$$W(r, r') = \sum_{k=0}^{N-1} x_k^{(rr')} (s_r s_{r'}^*)^k, \tag{2}$$

where * denotes complex conjugation. Assuming $x_0^{(rr')} \neq 0$, we can always set it equal to unity. Reality of the weights implies that $x_k^{(rr')} = x_{N-k}^{(rr')}$, which means that one can describe them by just $[N/2]$ real parameters. The weights are invariant under the global group Z_N, $s_r \to \omega^k s_r$, as well as charge conjugation $C : s_r \to s_r^*$.

In what follows, except for the last section, we consider the graph to be a regular lattice, with translationally invariant couplings $x_k^{(rr')}$, which may however depend on the lattice orientation of the edge (rr').

For $N = 2$ and 3 it is easy to see that the model is equivalent to the Ising model and three state Potts model respectively, whose critical points correspond to well-understood minimal conformal field theories with $c < 1$. The higher values of N are related to more complicated theories with rich phase diagrams [18]. Nearest-neighbor Z_N models exhibit Kramers–Wannier duality symmetry, which is important in understanding their phase diagrams. If all of x_k with $k \geqslant 1$ have the same value, we have the well-known N-state Potts model, which has a single first-order transition for $N > 4$. In general, for $N > 3$ the theory has a critical surface on which the exponents may vary continuously. It has been conjectured that there are $n < \frac{1}{2}N$ points in the phase diagram which correspond to special kind of conformal field theories called Fateev–Zamolodchikov (FZ) parafermionic models of the first kind [5]. This has been verified by numerical simulations [18]. Our lattice holomorphic operators, described below, are another evidence for this conjecture. These conformal field theories have the central charge

$$c = \frac{2(N-1)}{N+2}, \tag{3}$$

the cases $N = 2, 3$ coinciding with the central charge of $p = 2, 5$ minimal CFTs. Kramers–Wannier symmetry manifests itself in $N - 1$ order parameter fields σ_m and $N - 1$ dual disorder fields μ_m, which are conjectured to be the continuum version of the order and disorder variables on the original lattice. Zamolodchikov and Fateev, using only the Z_N symmetry, showed that there also exist holomorphic operators (that is, operators whose correlation functions are analytic functions except at coincident points) with the following dimensions (equal to their conformal spin):

$$p_m = \frac{m(N-m)}{N}, \qquad m = 1, \ldots, N-1. \tag{4}$$

We now give our construction of the lattice analogs of these objects. We first consider the homogeneous isotropic Z_N model on the square lattice.

In the first step let us define the disorder operators $\mu_{\tilde{r}}$ on the sites of dual lattice, which, in this case, is again another square lattice whose vertices are at the centers $\tilde{r}$ of the faces of the original lattice. The insertion of a disorder operator $\mu_{\tilde{r}m}$ corresponds to modifying the weights so that the order operator s_r has monodromy $s_r \to \omega^{-m} s_r$ on taking the point r in a closed circuit around $\tilde{r}$. This is equivalent to introducing a path, or string, on the sites of the dual lattice from $\tilde{r}$ to infinity (or some other point on the boundary), such that the weights on edges (rr') intersected by the string are modified by the substitution $s_r s_{r'}^* \to s_r \omega^{-m} s_{r'}^*$.

Thus, the disorder operator has the following form for the general Z_N model:

$$\mu_{\tilde{r}m} = \prod_{(rr')\ \text{intersected by string}} \frac{\sum_{k=0}^{N-1} x_k (s_r \omega^{-m} s_{r'}^*)^k}{\sum_{k=0}^{N-1} x_k (s_r s_{r'}^*)^k}. \tag{5}$$

It is not difficult to see that the disorder variables have the same C-symmetry similar to the spin variables, $\mu_m = \mu_{N-m}$, and that, up to a gauge transformation, the definition (5) is path independent.

Consider the square whose vertices are made by the two neighboring spin variables s_1, s_2 and the two neighboring disorder variables $\mu_{\tilde{1}}, \mu_{\tilde{2}}$ (figure 1). By taking the string from $\tilde{1}$ to run along the dual edge $(\tilde{1}\tilde{2})$, from (5) we can write the following relation:

$$\mu_{\tilde{1}m} = \frac{\sum_{k=0}^{N-1} x_k (s_1 \omega^{-m} s_2^*)^k}{\sum_{k=0}^{N-1} x_k (s_1 s_2^*)^k} \mu_{\tilde{2}m}. \tag{6}$$

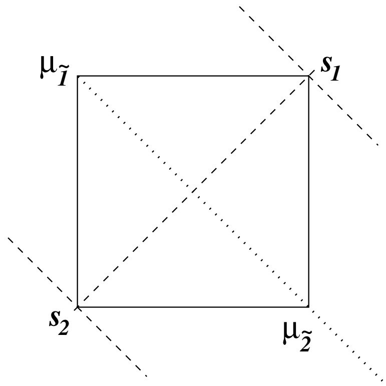

Figure 1. An elementary square of the covering lattice (solid lines—rotated by 45° to correspond to the conventions in the text). The opposite pairs of vertices are associated with order (s) and disorder (μ) operators, respectively. The parafermions ψ are associated with the edges. Discrete holomorphicity at the FZ points means that the contour sum of the parafermions around each face vanishes. Also shown is part of the original square lattice (dashed lines) and the string (dotted line) attached to the disorder operators.

We will need the exact value of x_k at the FZ point, which for the isotropic square lattice have the following compact form [4]:

$$x_{ck} = \prod_{j=0}^{k-1} \frac{\sin\left(\frac{\pi j}{N} + \frac{\pi}{4N}\right)}{\sin\left(\frac{\pi(j+1)}{N} - \frac{\pi}{4N}\right)}. \tag{7}$$

For clarity, let us first focus on the simplest case $N = 2$. By multiplying both side of (6) by the denominator and then by multiplying the result by s_1 we find the following equation:

$$s_1\mu_{\tilde{1}} + x_c s_2\mu_{\tilde{1}} = s_1\mu_{\tilde{2}} - x_c s_2\mu_{\tilde{2}}. \tag{8}$$

A similar equation can be found by exchanging $s_1 \leftrightarrow s_2$. By combining these two equations (multiplying the second equation by $-\mathrm{i}$ and adding this to the first) one finds

$$-\mathrm{e}^{\frac{\mathrm{i}\pi}{2}} s_1\mu_{\tilde{1}} - \mathrm{i}\,\mathrm{e}^{-\frac{\mathrm{i}\pi}{4}} s_2\mu_{\tilde{1}} + s_2\mu_{\tilde{2}} + \mathrm{i}\,\mathrm{e}^{\frac{\mathrm{i}\pi}{4}} s_1\mu_{\tilde{2}} = 0. \tag{9}$$

Note that for this to happen it is crucial that we use the critical value $x_{c1} = \tan(\pi/8) = \sqrt{2}-1$. (For the special case of the Ising model, when $x \neq x_c$, the right-hand side of (9) is proportional to the antiholomorphic fermion [9].)

Equation (9) has the form of a discrete contour integral around each elementary square of the covering lattice (the union of the dual lattice with the original lattice):

$$\sum_e \psi_e \delta z_e = 0 \qquad \text{with} \qquad \psi_{r\tilde{r}} = e^{\frac{-i\theta_{r\tilde{r}}}{2}} s_r\mu_{\tilde{r}}, \tag{10}$$

where $\theta_{r\tilde{r}}$ is the angle that the directed segment $r\tilde{r}$ makes with the x-axes, with the convention $-\pi \leqslant \theta_{r\tilde{r}} < \pi$. In this case these angles are just $-\pi, -\frac{\pi}{2}, 0, \frac{\pi}{2}$ but in the following section we will see more general cases. By changing $\mathrm{i} \to -\mathrm{i}$ one can find the similar equation for the discrete antiholomorphic fermions. These two linear equations are a lattice discretized form of the Cauchy–Riemann equations.

The calculation for $N = 3$ is very similar. Multiplying both sides of equation (6) by the denominator, and then by s_1, $s_1^2 s_2^*$ and s_2 we can find three linear equations in the six variables

$s_1\mu_{\tilde{1},\tilde{2}}, s_1^2 s_2^* \mu_{\tilde{1},\tilde{2}}, s_2\mu_{\tilde{1},\tilde{2}}$. By eliminating $s_1^2 s_2^* \mu_{\tilde{1},\tilde{2}}$ one again finds a discretely holomorphic equation of the form (10) with now

$$\psi_{r\tilde{r}} = \mathrm{e}^{\frac{-2\mathrm{i}\theta_{r\tilde{r}}}{3}} s_r \mu_{\tilde{r}}. \tag{11}$$

Once again, this works only at the FZ point given by (7).

The case $N = 4$ is very similar and we will discuss the results later. $N = 5$ is more interesting. This model has two FZ points $(x_1, x_2) = (x_{c1}, x_{c2})$ and (x_{c2}, x_{c1}) corresponding to the transformation $s_r \to s_r^2$ which exchanges x_1 and x_2. For this case equation (6) has the following form:

$$\begin{aligned}\mu_{\tilde{1}m}&(1 + x_{c1}s_1s_2^* + x_{c2}(s_1s_2^*)^2 + x_{c2}(s_1s_2^*)^3 + x_{c1}(s_1s_2^*)^4)\\ &= \mu_{\tilde{2}m}\big(1 + \mathrm{e}^{-\frac{2\pi im}{5}} x_{c1}s_1s_2^* + \mathrm{e}^{-\frac{4\pi im}{5}} x_{c2}(s_1s_2^*)^2\\ &\quad + \mathrm{e}^{-\frac{6\pi im}{5}} x_{c2}(s_1s_2^*)^3 + \mathrm{e}^{-\frac{8\pi im}{5}} x_{c1}(s_1s_2^*)^4\big).\end{aligned} \tag{12}$$

For the $m = 1$ case by multiplying the above equation by $s_1, s_2, s_1^2s_2^*, s_1^3s_2^{*2}, s_1^4s_2^{*3}$ one finds the following equation

$$\begin{aligned}\mu_{\tilde{1}1}&\big(s_1 + x_{c1}s_1^2s_2^* + x_{c2}s_1^3s_2^{*2} + x_{c2}s_1^4s_2^{*3} + x_{c1}s_2^{*4}\big)\\ &= \mu_{\tilde{2}1}\big(s_1 + \mathrm{e}^{-\frac{2\pi\mathrm{i}}{5}} x_{c1}s_1^2s_2^* + \mathrm{e}^{-\frac{4\pi\mathrm{i}}{5}} x_{c2}s_1^3s_2^{*2}\\ &\quad + \mathrm{e}^{-\frac{6\pi\mathrm{i}}{5}} x_{c2}s_1^4s_2^{*3} + \mathrm{e}^{-\frac{8\pi\mathrm{i}}{5}} x_{c1}s_2\big),\end{aligned} \tag{13}$$

and four other equations by permuting the terms $s_1, s_1^2s_2^*, s_1^3s_2^{*2}, s_1^4s_2^{*3}, s_2$. The next step is to eliminate the terms involving $s_1^2s_2^*, s_1^3s_2^{*2}, s_1^4s_2^{*3}$ from the above five equations. The calculation is rather cumbersome, so we resorted to using Mathematica. The result is as follows: for $m = 1$ it is possible to eliminate the above terms only at the first FZ point (x_{c1}, x_{c2}), in which case we obtain the holomorphic equation (10) with the holomorphic variable $\psi_{r\tilde{r}}^1 = \mathrm{e}^{-\frac{4\mathrm{i}\theta_{r\tilde{r}}}{5}} s_r\mu_{\tilde{r}1}$.

For $m = 2$ we first multiply equation (12) by $s_1^2, s_1^3s_2^*, s_1^4s_2^{*2}, s_2^{*3}, s_1s_2^{*4}$ to find

$$\begin{aligned}\mu_{\tilde{1}2}&\big(s_1^2 + x_{c1}s_1^3s_2^* + x_{c2}s_1^4s_2^{*2} + x_{c2}s_2^{*3} + x_{c1}s_1s_2^{*4}\big)\\ &= \mu_{\tilde{2}2}\big(s_1^2 + \mathrm{e}^{-\frac{4\pi\mathrm{i}}{5}} x_{c1}s_1^3s_2^* + \mathrm{e}^{-\frac{8\pi\mathrm{i}}{5}} x_{c2}s_1^4s_2^{*2}\\ &\quad + \mathrm{e}^{-\frac{12\pi\mathrm{i}}{5}} x_{c2}s_2^{*3} + \mathrm{e}^{-\frac{16\pi\mathrm{i}}{5}} x_{c1}s_1s_2^{*4}\big),\end{aligned} \tag{14}$$

and four other equations by permuting the different terms. Again we should eliminate the terms involving variables $s_1^3s_2^*, s_1^4s_2^{*2}, s_1s_2^{*4}$. In this case, we can find a holomorphic equation only at the second FZ point $(x_1, x_2) = (x_{c2}, x_{c1})$ with the holomorphic variable $\psi_{r\tilde{r}}^2 = \mathrm{e}^{\frac{-6\mathrm{i}\theta_{r\tilde{r}}}{5}} s_r^2\mu_{\tilde{r}2}$.

The calculation for $m = 3$ and $m = 4$ is similar and gives the antiholomorphic parafermions.

The generalization to higher values of N is straightforward. By simplifying equation (6) at the FZ points we conjecture that the following variables will be discretely holomorphic:

$$\psi_{r\tilde{r}}^m = \mathrm{e}^{-\mathrm{i}p_m\theta_{r\tilde{r}}} s_r^m \mu_{\tilde{r}m}, \tag{15}$$

for $1 \leqslant m \leqslant [N/2]$. The same expression with the argument of the exponential having the opposite sign is discretely antiholomorphic for $[N/2] \leqslant m \leqslant N-1$. (This is just the complex conjugate of $\psi_{r\tilde{r}}^{N-m}$.) We checked the conjecture for $N = 6$. In this case, we find all of the holomorphic operators at the one FZ point (x_{c1}, x_{c2}, x_{c3}). These discretely holomorphic quantities are the candidates for the holomorphic parafermions of FZ parafermionic CFT in the continuum limit. We should emphasize that these operators are holomorphic only at the FZ points and not elsewhere on the critical self -dual surfaces. For example, for $N = 4$, $\psi_{r\tilde{r}}^1$ is not holomorphic at the critical point of the four-state Potts model. However $\psi_{r\tilde{r}}^2$, which has conformal spin 1, is holomorphic all the way along the critical line $2x_{c1} + x_{c2} = 1$.

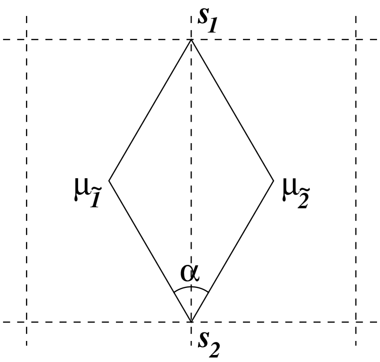

Figure 2. Embedding of the anisotropic square lattice (dashed lines) in the plane. The faces of the covering lattice form rhombi with opening angles α (shown) and $\pi - \alpha$.

We should also note another curious feature, which is most simply illustrated for the Ising case. Although we have shown that the linear relation (9) implies the discrete holomorphicity relation (10) for the parafermion $\psi_{r\tilde{r}} = \mathrm{e}^{-\frac{\mathrm{i}\theta_{r\tilde{r}}}{2}} s_r \mu_{\tilde{r}}$, if we define another quantity $\hat{\psi}_{r\tilde{r}} \equiv \mathrm{e}^{\frac{3\mathrm{i}\theta_{r\tilde{r}}}{2}} s_r \mu_{\tilde{r}}$, then (9) also implies that $\sum_e \hat{\psi}_e \delta z_e^* = 0$, that is $\hat{\psi}_{r\tilde{r}}$ is discretely antiholomorphic. This is quite general, and holds for other values of N and the more general lattices discussed in the following section: if $\psi_{r\tilde{r}}$ is discretely holomorphic, then $\hat{\psi}_{r\tilde{r}} = \mathrm{e}^{2\mathrm{i}\theta_{r\tilde{r}}} \psi_{r\tilde{r}}$ is discretely antiholomorphic.

3. Z_N model on other lattices

In the previous section we found additional evidence that the Zamolodchikov–Fateev parafermionic CFT is a good candidate for the continuum limit of the FZ point of Z_N models on the square lattice, by identifying discretely holomorphic parafermions on the lattice. Note this construction was possible only at the FZ points. One may therefore try to invert the argument and locate the FZ point by requiring discrete holomorphicity. Since, by universality, the same conformal field theory should describe the continuum limit of suitable Z_N models on other lattices, one would expect to be able to identify discretely holomorphic objects in this case, and thereby deduce the location of the FZ points. In fact we will show that this is possible in many cases and that the critical weights of the models are just related to the geometry of the covering lattice when it is suitably embedded in the plane. Note that the notion of holomorphicity is not invariant under a general diffeomorphism of the plane, only under conformal transformations. Therefore we expect that the identification of discretely holomorphic objects will depend on choosing a particular embedding of the lattice into the plane, modulo conformal mappings. In what follows we study this problem for the anisotropic square, honeycomb and triangular lattices, and then the more general case of a 'Baxter lattice'.

Let us first investigate the square lattice with unequal weights x_k^x, x_k^y in the x- and y-directions, respectively. In this case, it is clear that in order to maintain invariance under lattice translations and reflection symmetry about the x- and y-axes the only transformations allowed are relative scalings of the x- and y-coordinates [19]. In this case, each elementary square of the covering lattice is deformed into a rhombus (figure 2).

Defining the parafermions by (15), we can try to demand that the discrete contour integral around each elementary rhombus vanishes as before. As an example, we show the case of the

Ising model in detail. Taking an arbitrary linear combination of (8) and its counterpart with $s_1 \leftrightarrow s_2$ we have

$$\frac{a + x_1^y}{a x_1^y - 1} s_1 \mu_{\bar{1}} + \frac{a x_1^y + 1}{a x_1^y - 1} s_2 \mu_{\bar{1}} + \frac{a - x_1^y}{1 - a x_1^y} s_1 \mu_{\bar{2}} + s_2 \mu_{\bar{2}} = 0. \tag{16}$$

On the other hand, for the rhombus in figure 2 the contour sum is

$$-\psi_{1\bar{1}} + \mathrm{e}^{-\mathrm{i}(\pi-\alpha)} \psi_{2\bar{1}} + \psi_{2\bar{2}} + \mathrm{e}^{\mathrm{i}\alpha} \psi_{1\bar{2}} = 0 \tag{17}$$

where $\psi_{1\bar{1}} = \mathrm{e}^{\mathrm{i}\pi/2} s_1 \mu_{\bar{1}}$, $\psi_{2\bar{1}} = \mathrm{e}^{-\mathrm{i}\alpha/2} s_2 \mu_{\bar{1}}$, $\psi_{2\bar{2}} = s_2 \mu_{\bar{2}}$ and $\psi_{1\bar{2}} = \mathrm{e}^{\mathrm{i}(\pi-\alpha)/2} s_1 \mu_{\bar{2}}$. Comparing equations (16) and (17) we find that they are consistent only at a particular value of a and for $x_1^y = \tan(\alpha/4)$. Similarly, we find that the contour sums around the rhombi aligned in the other direction can vanish only if $x_1^x = \tan((\pi - \alpha)/4)$.

Note that the comparison of (16), (17) implies three complex equations. If instead of choosing the elementary face of the covering lattice to be a rhombus we take an arbitrary quadrilateral with edges δz_e, where $\sum_{e=1}^{4} \delta z_e = 0$, this would introduce two further complex unknowns (ratios of δz_e) into (17), besides the unknown a. Given x_1^y, the three complex equations therefore determine a and the shape of the quadrilateral. We have already shown that a rhombus with a suitable opening angle satisfies all the equations. Therefore, one would expect that they cannot be satisfied for any other quadrilateral with unequal edge lengths. This can be checked explicitly.

For higher values of N we find that the contour sum around the rhombi vanishes as long as the weights satisfy

$$x_k^y = x_{ck}(\alpha) \equiv \prod_{i=0}^{k-1} \frac{\sin\left(\frac{\pi i}{N} + \frac{\alpha}{2N}\right)}{\sin\left(\frac{\pi(i+1)}{N} - \frac{\alpha}{2N}\right)} \qquad x_k^x(\alpha) = x_{ck}(\pi - \alpha). \tag{18}$$

The result agrees with that found by Fateev and Zamolodchikov [4] by imposing the star-triangle relations.

Next we consider the isotropic honeycomb and triangular lattices, which are mutually dual with the vertices of the honeycomb lattice at the centers of the faces of the triangular lattice and vice versa. The elementary cells of the covering lattice give a regular rhombus tiling of the plane, with $\alpha = \frac{2\pi}{3}$ for the honeycomb lattice and $\alpha = \frac{\pi}{3}$ for the triangular lattice. The mathematics then proceeds just as for the anisotropic square lattice above, with the result that one can find discretely holomorphic parafermions as long as the weights satisfy (18) with the appropriate values of α. This we then conjecture to correspond to the FZ point(s) for these lattices.

A more interesting case is that of the homogeneous triangular lattice with unequal weights $x_k^{(1)}$, $x_k^{(2)}$ and $x_k^{(3)}$ in the three lattice directions. In this case, the question of where to locate the dual vertices is crucial. On the basis of the above observations, we adopt the following construction. Consider a (proper) embedding of the regular triangular lattice into the plane by some general linear transformation of the coordinates. This gives a regular tiling of the plane by triangles. For each triangular face with Z_N spins at the vertices, locate the dual vertex at the *circumcenter*, that is the point at which the three perpendicular bisectors of the edges meet, equidistant from the three vertices. This construction guarantees that adjacent pairs of vertices and dual vertices always form a rhombus. Each triangle is associated with three different rhombi with angles α_1, α_2 and $\alpha_3 = \pi - \alpha_1 - \alpha_2$, see figure 3. If now we choose the weights so that

$$x_{1k} = x_{ck}(\alpha_1), \qquad x_{2k} = x_{ck}(\alpha_2), \qquad x_{3k} = x_{ck}(\pi - \alpha_1 - \alpha_2), \tag{19}$$

it follows from our general analysis that we then can identify discretely holomorphic parafermions. Transforming back to the original regular triangular lattice with unequal

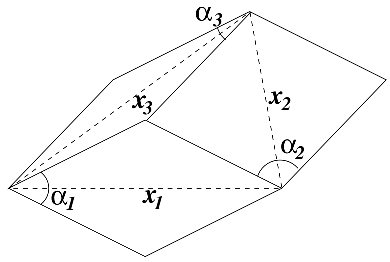

Figure 3. Embedding of one face of a regular anisotropic triangular lattice (dashed lines) in the plane. The faces of the covering lattice are rhombi with opening angles α_1, α_2 and $\alpha_3 = \pi - \alpha_1 - \alpha_2$.

weights, we therefore conjecture that (19) gives the general critical surface of the FZ point(s). Note that it is possible for one of the α_j to be negative, if the largest angle of the deformed triangle is obtuse, corresponding to the interactions in one of the lattice directions being (weakly) antiferromagnetic. (However these points are still on the ferromagnetic critical surface, and have the Fateev–Zamolodchikov CFT [5] as their conjectured scaling limit.) On transforming back to the regular lattice, the images of the dual vertices do not in general lie at the centers of the triangular faces (in fact they may lie outside the face).

It is easy to check that these results are consistent with previously known results for the $N = 2$ Ising model and $N = 3$ three states Potts model [6, 20].

The FZ critical points for the honeycomb lattice with anisotropic couplings follow directly by duality from those of the triangular lattice, which corresponds to letting $\alpha_j \to \pi - \alpha_j$ in (19).

We now indicate how our results extend to a more general kind of inhomogeneous lattice, called a Z-invariant or Baxter lattice [6]. This is a planar graph $\mathcal{L}$ which is a union of M simple (non-self-intersecting) curves crossing the complex plane from $x_j - i\infty$ to $x'_j + i\infty$, where the values $\{x_j\}$ and $\{x'_j\}$ ($1 \leqslant j \leqslant M$) are distinct, and with the further property that a given curve can intersect any of the others at most once (figure 4). The faces of $\mathcal{L}$ are 2-colorable, and in general we can assume that its vertices are all of degree 4 (if not we deform the curves slightly so this is true). Consider now the planar graph $\mathcal{G}$ whose vertices are associated with each black face of $\mathcal{L}$ and whose edges E pass through the vertices of $\mathcal{L}$. We can define a Z_N model on $\mathcal{G}$ with general weights $x_k^{(E)}$. The vertices of the dual lattice $\mathcal{G}^*$ then correspond to the white faces of $\mathcal{L}$. The vertices of the covering graph $\mathcal{C}$ (the union of the vertices of $\mathcal{G}$ and $\mathcal{G}^*$) each correspond to a face of $\mathcal{L}$, irrespective of color. Note that all of the faces of $\mathcal{C}$ have degree 4. We are free to embed this lattice in the plane in any way we choose, just as for the anisotropic triangular lattice earlier. However, a remarkable theorem due to Kenyon and Schlenker [7] states that (in the case considered here, where the original curves do not self-intersect and cross any other at most once) there is a *rhombic embedding* of $\mathcal{C}$ into the plane, that is, one in which all the edges have equal length. Each rhombus corresponds to an edge E of $\mathcal{G}$ and defines an opening angle α_E. As before, we may define parafermions on the edges of $\mathcal{C}$ and demand that they be discretely holomorphic when summed around the edges of each rhombus. This will be the case if we choose $x_k(E) = x_{kc}(\alpha_E)$, as given by (18). We conjecture that these values give the location of the FZ points on this inhomogeneous model. The rhombic embedding specifies how this model should be embedded in the plane in order that its scaling limit be given by the FZ conformal field theory.

Finally, we discuss how the conditions that allow the Yang–Baxter (star-triangle) relations in these models are compatible with discrete holomorphicity. Consider part of a lattice $\mathcal{L}$ where

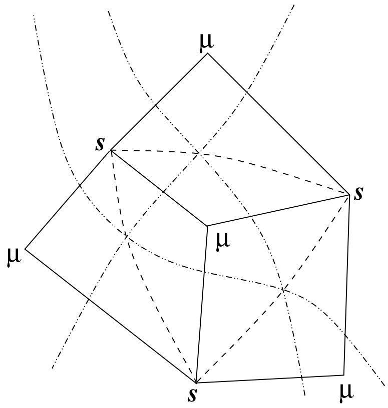

Figure 4. Part of a Baxter lattice. The faces of the graph $\mathcal{L}$ formed by the curved lines are 2-colorable (not shown). Order variables s and disorder variables μ are associated with alternately colored faces, respectively. The covering lattice is shown as solid lines. The theorem of Kenyon and Schlenker [7] asserts that for every such graph the covering lattice admits a rhombic embedding in the plane, that is one where all its edges have the same length.

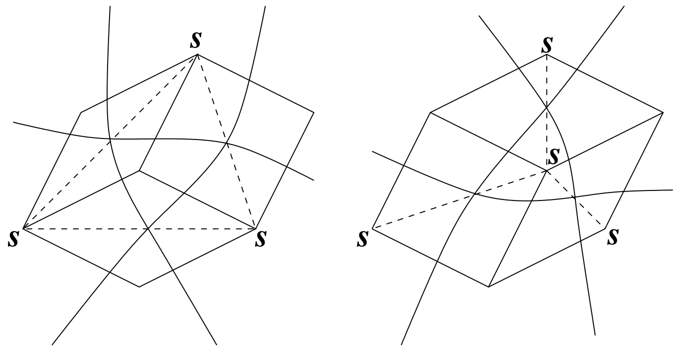

Figure 5. Two different tilings of a hexagon by the same set of three rhombi. The right-hand case has, in the graph $\mathcal{G}$, an additional vertex associated with an order operator s as compared to that on the right. Discrete holomorphicity for each rhombus fixes the couplings on the dashed lines to be related by the star-triangle transformation. The two pictures are also related in the original graph $\mathcal{L}$ by moving one of the curves past the vertex formed by the other two—the Yang–Baxter relation.

three lines meet at a point. This can be resolved in two different ways (see figure 5). The Yang–Baxter relations guarantee that the Boltzmann weights, keeping all the other lines fixed, are independent of how this is done. If we 2-color the faces of $\mathcal{L}$, it can be seen that the second resolution adds one more black face, that is one more vertex of $\mathcal{G}$. The invariance under this is the star-triangle relation.

The rhombic embeddings in the two cases are shown in figure 5. It can be seen that in both cases the three rhombuses fit together to form a hexagon with opposite sides parallel and that they simply correspond to two different tilings of the hexagon by the same three rhombi. The condition that the contour sum of the parafermions around each rhombus should vanish implies that the couplings should satisfy $\tilde{x}_k = x_{kc}(\pi - \alpha)$, which are the critical star-triangle relations.

4. Conclusion and discussion

In this paper, we considered the nearest-neighbor Z_N model on some fairly general lattices. We identified the dual disorder operators and used these to define parafermionic operators which reside on the edges of the covering lattice. For the isotropic square, triangular and honeycomb lattices we showed that, at the Fateev–Zamolodchikov critical points, these parafermions obey the discrete version of the Cauchy–Riemann relations, that is their contour sum around each elementary face of the covering lattice vanishes. We then extended this idea to regular lattices with anisotropic couplings and showed that, if they are correctly embedded in the plane, it is possible once again to identify discretely holomorphic parafermions at particular values of the couplings. These we conjecture to correspond to the FZ points on these anisotropic lattice, which agrees with previously known cases.

A crucial feature of these embeddings is that the faces of the covering lattice should be rhombi. This enabled us to extend our results to more general inhomogeneous lattices of the Baxter type, thanks to a theorem [7] which guarantees the existence of a rhombic embedding in such cases. In this picture, the relation between discrete holomorphicity and the Yang–Baxter, or star-triangle, relations becomes particularly clear.

We expect that the discretely holomorphic lattice parafermions become the holomorphic parafermions of the conformal field theory, but this requires further assumptions. As pointed out in [8, 9], if we regard the discrete Cauchy–Riemann equations as a linear system, there are in general twice as many unknowns as equations (only in the Ising case, when the phases of the parafermions are not free, there are the correct numbers). Morera's theorem, which allows one to deduce that a function f defined on $\mathbb{R}^2$ is analytic if its contour integral around every closed contour vanishes, applies only if f is also assumed to be continuous. If we could prove that all correlators of our discrete parafermions become continuous functions in the scaling limit, this would be sufficient to show complex analyticity, at least with sufficiently smooth boundaries. However this step is highly non-trivial, as evidenced by the ambiguity in the interpretation of the linear relations discussed at the end of section 2. There we showed that they lead to the identification of discrete parafermions $\psi_{r\tilde{r}}$ and also $\hat{\psi}_{r\tilde{r}} = \mathrm{e}^{2\mathrm{i}\theta_{r\tilde{r}}}\psi_{r\tilde{r}}$. If they corresponded to conformal fields in the scaling limit, they would have conformal spins p_m and $p_m - 2$. However, if the correlators of ψ become continuous functions in the scaling limit, this cannot be true of those of $\hat{\psi}$ and vice versa.

If this problem can be overcome, we believe that our identification of suitable discretely holomorphic quantities should be the first step in showing that suitable defined curves in the Z_N have SLE as their scaling limit, as has been recently conjectured [10, 11]. In order to do this, following the ideas of Smirnov [8], however, it is necessary to identify these quantities with observables of these curves which are martingales of some discrete exploration process. Since the domain walls (or, equivalently, high-temperature graphs) of the Z_N model do not correspond directly to simple lattice curves (except for $N = 2$), there are a number of difficulties yet to be overcome in this program.

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